1.6-1.7a Leslie's age structured model and Perron-Frobenius

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Assumptions:

- Population is closed to migration
- Only females are modeled directly
- Divide population into m fixed age groups, where m is the last reproductive age.
- As $t \to t+1$, individuals age from $i \to i+1$
- Individuals in the same age group have the same reproduction rate.

Mathematically

- $x_i(t)$ = number of females in ith age group at time t.
- b_i = average number of newborn females produced by one female in the ith age group that survives their birth time interval.
- s_i = fraction of ith age group that survives until $(i+1)$th age.

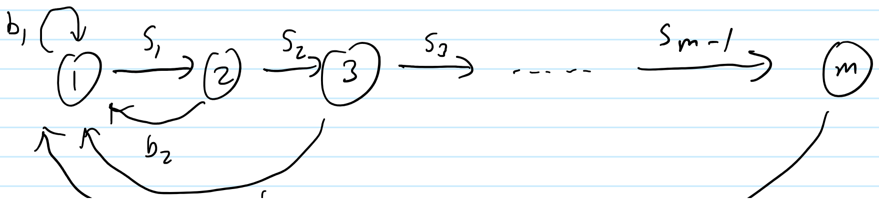
Then $x_1(t+1) = \sum_{i=1}^{m} b_i x_i(t)$ and $x_{i+1}(t+1) = s_i x_i(t)$, $i > 1$

Equivalently,

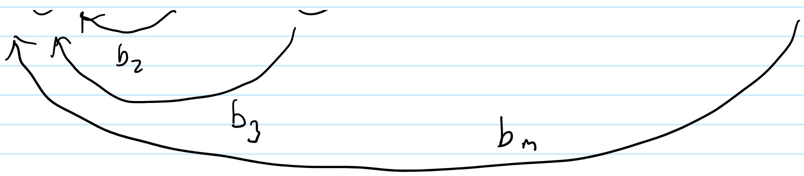
$$X(t+1) = \begin{pmatrix} x_1(t+1) \\ \vdots \\ \vdots \\ x_m(t+1) \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 & \cdots & \cdots & b_m \\ \hline s_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & s_{m-1} & 0 \end{pmatrix}}_{L = \text{Leslie matrix (aka projection matrix)}} \begin{pmatrix} x_1(t) \\ \vdots \\ \vdots \\ x_m(t) \end{pmatrix} = L X(t)$$

In general $X(t) = L^t X(0)$.

Life cycle graph



(directed graph or digraph)



| or digraph /

Def. A *permutation matrix* $P \in \mathbb{R}^{m \times m}$ has exactly one 1 in every row and col, and 0's everywhere else. Note: $P^T P = I$.

Def. 1.10 Let $A \in \mathbb{R}^{m \times m}$. A is *reducible* if $\exists$ permutation matrix P s.t.

$$P^T A P = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right],$$

where A_{11}, A_{22} are square matrices of nonzero size.

Otherwise, A is *irreducible*.

Def. Let $A \in \mathbb{R}^{m \times m}$ be a nonnegative matrix ($A \geq 0$). Then *digraph(A)* is the directed graph with m nodes labelled $\{1, \dots, m\}$ and edge $i \to j$ with weight a_{ji} if $a_{ji} \neq 0$.

Def. 1.11 If digraph(A) has a directed path from i to j $\forall\, i, j \in \{1, \dots, m\}$, then digraph$(A)$ is *strongly connected*.

Thm 1.2 digraph(A) is strongly connected iff A is irreducible.

Thm 1.3 (Frobenius) If $A \in \mathbb{R}^{m \times m}$ is irreducible and nonnegative ($A \geq 0$), then $\lambda = \rho(A)$ is a dominant eigenvalue of multiplicity 1. And, λ has a corresponding eigenvector with positive components.

Thm 1.4 (Perron) If $A \in \mathbb{R}^{m \times n}$ is positive, then $\lambda = \rho(A)$ is a strictly dom. eigenvalue of multiplicity 1. And, λ has a corresponding eigenvector with pos. components.

Def. 1.12 If $A \in \mathbb{R}^{m \times m}$ is irreducible and $A \geq 0$, and has h eigenvalues of maximum modulus, then A is *primitive* if $h = 1$ and *imprimitive* if $h \neq 1$.

h is the *index of imprimitivity*

and **imprimitive** if $h \neq 1$.

h is the **index of imprimitivity**

Thm 1.5 If $A \in \mathbb{R}^{m \times m}$ and $A \geq 0$, then A is primitive iff $A^p > 0$ for some $p \in \mathbb{Z}^+$.

Def. The **inherent net reproductive number R_0** is the expected number of offspring for an individual over its lifetime.

Ex. $R_0 = b_1 + b_2 s_1 + b_3 s_1 s_2 + \cdots + b_m s_1 \cdots s_{m-1}$ in the Leslie model.

Thm 1.6 Consider a Leslie matrix L. If L is irreducible and primitive, then $\lambda_1 = \rho(L) > 0$. Furthermore,

$$\lambda_1 = 1 \quad \text{iff} \quad R_0 = 1,$$
$$\lambda_1 > 1 \quad \text{iff} \quad R_0 > 1,$$
$$\lambda_1 < 1 \quad \text{iff} \quad R_0 < 1.$$

Additionally, we have a stable age distribution V_1,

$$V_1 = \begin{pmatrix} 1 \\ \frac{s_1}{\lambda_1} \\ \vdots \\ \frac{s_1 s_2 \cdots s_{m-1}}{\lambda_1^{m-1}} \end{pmatrix},$$

where V_1 is the eigenvector associated with λ_1.

Thm 1.7 An irreducible Leslie matrix L is primitive iff the birth rates satisfy the following relationship,

$$\underbrace{\text{g.c.d.}}_{\text{greatest common divisor}} \{ i \mid b_i > 0 \} = 1.$$